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THE CIRCLE NEAREST TO N GIVEN POINTS, AND THE POINT NEAREST TO N GIVEN CIRCLES.

By Julian Lowell Coolidge.

A. Suppose that we are given n points $(x_1, y_1)(x_2, y_2) \cdots (x_n, y_n)$. These points were expected to lie upon a circle, but, for some reason, they fail to do so. What circle shall we take as that which most nearly fulfills the condition of passing through them?

The problem, as so stated, is lacking in precision, owing to the obscurity of the words "most nearly." They may be given a precise meaning in various ways; if, however, we are at all under the sway of the dogma of least squares, it seems natural to state the problem exactly in the following terms:

To find a circle such that the sum of the squares of the distances from each of n given points to the nearest point of the circle shall be a minimum.[1]

Let (x, y) be the coördinates of the center of the required circle while r is its radius. Let us further assume, for simplicity of calculation, that the origin is at the center of gravity of the given points so that

$$\sum_i x_i = \sum_i y_i = 0 \qquad (i = 1, 2 \cdots n). \tag{1}$$

The condition imposed may be written

$$\Sigma[\,|\sqrt{(x - x_i)^2 + (y - y_i)^2}\,| - r]^2 = Minimum. \tag{2}$$

Let us write, for simplicity

$$|\sqrt{(x - x_i)^2 + (y - y_i)^2}\,| = d_i + r. \tag{3}$$

Equating to zero the three partial derivatives of (2) we have

$$\begin{aligned} \Sigma\, d_i &= 0, \\ \Sigma \frac{(x - x_i)d_i}{d_i + r} &= 0, \\ \Sigma \frac{(y - y_i)d_i}{d_i + r} &= 0. \end{aligned} \tag{4}$$

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[1] Vahlen "Konstruktionen und Approximationen," Leipzig, 1911, p. 126, says: "Für die konstruktive Ausgleichung . . . eines Kreises aus mehr als drei Punkten scheinen dem Bertotschen entsprechende einfache Verfahren noch nicht gefunden zu sein." For $n = 4$ he finds the circle whose *greatest* distance from any given point shall be a minimum.

THEOREM. *The circle sought is characterized by the fact that the geometric sum of the vectors from the given points to the respectively nearest points of the circle is zero, while the radius is the average of the distances from the given points to the center.*

It is immediately apparent that it would be quite hopeless to undertake to solve equations (3) and (4) in their present form. We can, however, reach an approximate solution by assuming that the points are nearly concyclic so that d_i is small compared with r, *i.e.*, that $(d_i/r)^2$ is negligible. Then

$$\frac{d_i}{d_i + r} = \frac{d_i}{r}\left(1 + \frac{d_i}{r}\right)^{-1} = \frac{d_i}{r},$$

$$\begin{aligned} \Sigma d_i = \Sigma(x - x_i)d_i = \Sigma(y - y_i)d_i &= 0, \\ \Sigma(x - x_i)^2 + \Sigma(y - y_i)^2 - nr^2 &= 0, \\ \Sigma(x - x_i)^3 + \Sigma(x - x_i)(y - y_i)^2 - r^2\Sigma(x - x_i) &= 0, \\ \Sigma(x - x_i)^2(y - y_i) + \Sigma(y - y_i)^3 - r^2\Sigma(y - y_i) &= 0. \end{aligned} \tag{5}$$

These equations are noteworthy for two reasons:

1. They are just the equations we should have reached, had we started from the assumption, which is nearly as natural as the one which we actually made, that the sum of the squares of the powers of the given points with regard to the circle sought should be a minimum.

2. Although apparently involving high powers of x and y they are quite easy to solve. Let us eliminate r^2 between the first and second, and between the first and third; we get

$$\begin{aligned} n\Sigma(x - x_i)^3 - \Sigma(x - x_i)^2\Sigma(x - x_i) + n\Sigma(x - x_i)(y - y_i)^2& \\ - \Sigma(x - x_i)\Sigma(y - y_i)^2 &= 0, \\ n\Sigma(y - y_i)^3 - \Sigma(x - x_i)\Sigma(y - y_i)^2 + n\Sigma(x - x_i)^2(y - y_i)& \\ - \Sigma(x - x_i)^2\Sigma(y - y_i) &= 0. \end{aligned} \tag{7}$$

These are reduced, with the aid of (1) to

$$\begin{aligned} 2(\Sigma x_i^2)x + 2(\Sigma x_i y_i)y &= \Sigma x_i(x_i^2 + y_i^2), \\ 2(\Sigma x_i y_i)x + 2(\Sigma y_i^2)y &= \Sigma y_i(x_i^2 + y_i^2). \end{aligned} \tag{6}$$

We thus reach our final solution

$$\begin{aligned} x &= \frac{\Sigma x_i(x_i^2 + y_i^2)\Sigma y_i^2 - \Sigma y_i(x_i^2 + y_i^2)\Sigma x_i y_i}{2[\Sigma x_i^2 \Sigma y_i^2 - (\Sigma x_i y_i)^2]}, \\ y &= \frac{\Sigma y_i(x_i^2 + y_i^2)\Sigma x_i^2 - \Sigma x_i(x_i^2 + y_i^2)\Sigma x_i y_i}{2[\Sigma x_i^2 \Sigma y_i^2 - (\Sigma x_i y_i)^2]}, \\ r &= \sqrt{\frac{\Sigma[(x - x_i)^2 + (y - y_i)^2}{n}}. \end{aligned} \tag{7}$$

B. Suppose that we are given n circles whose centers are the points $(x_1, y_2)(x_2, y_2) \cdots (x_n, y_n)$, while their respective radii are $r_1, r_2, \cdots, r_n$. These circles were supposed to pass through a common point, but failed to do so. What point shall we take to replace their supposed point of concurrence?

The first stages of this problem are surprisingly like those of *A*. If we seek a point such that the sum of the squares of its distances from the nearest points of the given circles shall be a minimum, the expression to minimize is

$$\Sigma(|\sqrt{(x - x_i)^2 + (y - y_i)^2}| - r_i)^2.$$

If we make a minute change of notation whereby

$$|\sqrt{(x - x_i)^2 + (y - y_i)^2}| = d_i + r_i; \qquad \Sigma\frac{x_i}{r_i^2} = \Sigma\frac{y_i}{r_i^2} = 0 \tag{8}$$

and equate to zero the partial derivatives with respect to x and y, we find

$$\begin{aligned} \Sigma\frac{(x - x_i)d_i}{d_i + r_i} &= 0, \\ \Sigma\frac{(y - y_i)d_i}{d_i + r_i} &= 0. \end{aligned} \tag{9}$$

THEOREM. *The point sought is characterized by the fact that the geometric sum of the vectors from there to the nearest points of the given circles is zero.*

Let us look for an approximate solution, following our previous work step by step we reach

$$\begin{aligned} \Sigma\frac{(x - x_i)^3}{r_i^2} + \Sigma\frac{(x - x_i)(y - y_i)^2}{r_i^2} - \Sigma(x - x_i) &= 0, \\ \Sigma\frac{(x - x_i)^2(y - y_i)}{r_i^2} + \Sigma\frac{(y - y_i)^3}{r_i^2} - \Sigma(y - y_i) &= 0. \end{aligned} \tag{10}$$

It would be pleasant to follow copy from here right to the end; unfortunately we can not do so, owing to the fact that we have nothing to correspond to the first equation (5). We may, however, rewrite (10) in the form

$$\Sigma[(x^2 + y^2) - 2(xx_i + yy_i) + (x_i^2 + y_i^2 - r_i^2)]\left(\frac{x - x_i}{r_i^2}\right) = 0,$$

$$\Sigma[(x^2 + y^2) - 2(xx_i + yy_i) + (x_i^2 + y_i^2 - r_i^2)]\left(\frac{y - y_i}{r_i^2}\right) = 0.$$

If we put

$$x^2 + y^2 = z$$

and remember (8), we easily find equations of the type

$$x = \frac{Az + B}{Pz^2 + Qz + R}, \qquad y = \frac{Cz + D}{Pz^2 + Qz + R}.$$

Here z is a root of a quintic equation with known coefficients. At this point the geometer steps lightly aside, leaving the quintic in possession of the field.

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